Computing Elastic-Rebound-Motivated Earthquake Probabilities in Un-segmented Fault Models – A New Methodology Supported by Physics-based Simulators

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Abstract  A methodology is presented for computing elastic-rebound-based probabilities in an un-segmented fault or fault system. The approach is less biased and more self-consistent than a logical extension of that applied most recently to multi-segment ruptures in California. It also enables the application of magnitude-dependent aperiodicity values, which the previous approach does not. Monte Carlo simulations are used to analyze long-term system behavior, which is generally found to be consistent with that of physics-based earthquake simulators. Results cast doubt that recurrence-interval distributions at points on faults look anything like traditionally applied renewal models, a fact that should be considered when interpreting paleoseismic data. We avoid such assumptions by changing the “probability of what” question. The new methodology is simple, although not perfect. It represents a reasonable way to represent first-order elastic-rebound predictability, assuming it is there in the first place, and for a system that clearly exhibits other complexities like aftershock triggering.

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**Introduction**

The primary scientific basis for computing long-term earthquake probabilities on a fault has been Reid’s elastic-rebound theory (Reid, 1911), which posits that the likelihood of a large earthquake goes down when and where there has been such an event, and increases with time as regional tectonic stresses rebuild. For example, all previous Working Groups on California Earthquake Probabilities (WGCEP, 1988, 1990, 1995, 2003, and 2007), which were commissioned for official policy purposes, have applied some variant of this model (e.g., see Field (2007) for a review of those through 2003).

For strictly segmented fault models, where earthquakes are assumed to always rupture the exact same “characteristic” fault segment, it is straightforward to use a point-process renewal model to compute elastic-rebound-based probabilities (e.g., Lindh, 1983). For example, the 1988 and 1990 WGCEPs did so using the Lognormal distribution, where average segment recurrence intervals were estimated from either observed inter-event times (e.g., Parkfield) or from average slip-per-event divided by fault slip rate.

If $f(t)$ represents the probability density function of recurrence intervals for the chosen renewal model, where $t$ is relative to the date of last earthquake, then the probability of having an event in the next $\Delta T$ years (the forecast duration), conditioned by the fact that it has been $T$ years since the last event, is

$$P(T \leq t \leq T + \Delta T | t > T) = \frac{\int_T^{T+\Delta T} f(t) dt}{\int_T^{\infty} f(t) dt}.$$ 

**Figure 1** illustrates this calculation for the Lognormal distribution using an example from WGCEP 2008.
**Figure 1.** Illustration of a conditional probability calculation using the Lognormal distribution, and parameters from WGCEP 2008 for the Cholame segment of the San Andreas Fault. The conditional 30-year probability ($P_{30}$) is equal to the area of the darker shaded region divided by that of the total shaded region.

**Figure 2** compares the Lognormal distribution with some other renewal models that have been proposed for earthquake probability calculations. These include the Brownian Passage Time (BPT) model (Matthews et al., 2002; WGCEP 2003 and 2007), the Weibull distribution (Rundle et al., 2006), and the Exponential model (the latter being the time-independent model used when date-of-last-event information is lacking). Note that for ease of comparison, we parameterize all of these distributions in terms of the mean ($\mu$) and aperiodicity ($\alpha$) of recurrence intervals, where aperiodicity is defined as the standard deviation divided by the mean (the expected coefficient of variation, COV). The relationships of $\mu$ and $\alpha$ to the more customary parameterizations of each renewal model are given in the Appendix of this paper. Conditional probabilities depend on the mean recurrence interval ($\mu$), the aperiodicity ($\alpha$), time since last event ($T$), and the forecast duration ($\Delta T$). However, scaling properties reduce these to only three independent parameters: aperiodicity ($\alpha$), time since last event normalized by the mean ($T/\mu$), and forecast duration normalized by the mean ($\Delta T/\mu$). Below, the latter two are referred to as “normalized time since last event” and “normalize duration”, respectively.
Figure 2. Probability-density function (PDF) of recurrence intervals (top), and the 1-year conditional probability of occurrence as a function of time since last event (bottom), for various renewal models as labeled. The mean recurrence interval is 100 years for all examples, and two aperiodicity values are illustrated ($\alpha = 0.2$ and $\alpha = 0.6$). Note that the Lognorma, $\alpha = 0.2$ PDF is completely hidden behind the BPT, $\alpha = 0.2$ PDF (the two are visually identical).

As discussed by WGCEP 2007, time-dependent probability calculations are not straightforward when strict segmentation assumptions are relaxed. In fact, the algorithm used by the two most recent WGCEPs (2003 and 2007) exhibits a self-consistency problem that will be discussed in the next section, and those applied earlier have even bigger issues (see Appendix N of the UCERF2 report by Field and Gupta (2007)). Because a primary goal for the next Uniform California Earthquake Rupture Forecast (version 3, or UCERF3) has been to relax segmentation (Field et al., 2013), we have needed a more self-consistent methodology for implementing elastic
rebound. We present such a methodology here, after reviewing the previous approach and evaluating what physics-based simulators imply with respect to elastic-rebound predictability. To avoid ambiguity, we reserve the term “segment” to indicate model-imposed limits on rupture endpoints (as in UCERF2) whereas fault “section” is used when no such impositions are implied (e.g., UCERF3).

All time-dependent methodologies discussed here assume the availability of a long-term earthquake-rate model, which gives the frequency \( f_r \) of each \( r^{th} \) large rupture on a fault or fault system, where the latter is represented by some number of fault sections or segments (identified with index \( s \)). For example, WGCEP (2003) divided faults into a few, relatively long segments (e.g., just four for the Northern San Andreas Fault), and derived the long-term rate of single and multi-segment ruptures via slip-rate constraints and expert opinion. For UCERF3, segmentation has effectively been relaxed by subdividing faults into many more, shorter-length subsections (Figure 3), and multi-fault ruptures are included between all sections that pass some basic plausibility criteria (Milner et al., 2013). The long-term rate of all supra-seismogenic (≥M6.5) ruptures in the fault system is then solved for using a formal inversion that satisfies various data constraints (Field et al., 2013; Page et al., 2013).

Given a long-term earthquake rate model, the frequency of large events on each fault-section can easily be computed by summing the rates of ruptures:

\[
f_s = \sum_{r=1}^{R} G_{sr} f_r
\]

where \( f_s \) is the frequency of the \( s^{th} \) section (or subsection), and matrix \( G_{sr} \) indicates which sections are utilized by each rupture (values are 1 or 0). The mean recurrence interval of each section is then computed as

\[
\mu_s = \frac{1}{f_s}
\]
Figure 3. **a)** One of the two fault models (FM 3.1) utilized in the UCERF3 long-term model, composed of 2606 fault subsections, each of which has a length that is approximately equal to half the down-dip width. The UCERF3 long-term model quantifies the rate of 253,706 possible ruptures within this fault system. **b)** The fault model used in the physics-based simulators described in the text, where each fault has been divided into 3 by 3 km “elements”. **NEED TO IMPROVE THE BOTTOM IMAGE**

**UCERF2 Methodology**

Elastic-rebound probabilities for UCERF2 were computed using a methodology developed originally by WGCEP (2003). The first step is to compute the BPT-based probability that each
section will rupture, \( p_s^{BPT} \), using the average section recurrence interval given above \( (\mu_s) \), the years since last event \( (T_s) \), an assumed aperiodicity \( (\alpha) \), and the forecast duration \( (\Delta T) \). The conditional probability of the \( r^{th} \) rupture is then computed from associated section probabilities as

\[
P_r^{U2} = f_r \frac{\sum (p_s^{BPT} \cdot M_{os} / f_s)}{\sum M_{os}}
\]

where \( f_r \) and \( f_s \) are from the long-term model as discussed above, \( M_{os} \) is the long-term moment rate of the \( s^{th} \) section (the product of slip rate, area, and shear rigidity), and the sums are only over ruptures that utilize the \( s^{th} \) section (we have dropped the \( G_{sr} \) matrix for simplicity here). The “\( U2 \)” superscript indicates that this formula was used most recently in UCERF2, even though it originated with WGCEP (2003). Note also that equation (1) was originally obtained from the FORTRAN code of WGCEP (2003), as it is only described with words in their report. We now modify this equation to make it a little more intuitive.

If the expected long-term number of occurrences on each fault section over the forecast duration is small \((f_s \Delta T \ll 0.1)\), then \( p_s^{Pois} \approx f_s \Delta T \), where \( p_s^{Pois} \) is the Poisson probability of one or more events on the section \((p_s^{Pois} = 1 - e^{-f_s \Delta T})\). Under this condition, we can substitute \((p_s^{Pois} / \Delta T)\) for \( f_s \), and likewise with respect to substituting \( p_r^{Pois} / \Delta T \) for \( f_r \) (since rupture rates are even lower than section rates), which transforms the above equation into

\[
P_r^{U2} \approx p_r^{Pois} \frac{\sum M_{os} (p_s^{BPT} / p_s^{Pois})}{\sum M_{os}}
\]

We can now see that UCERF2 rupture probabilities are effectively the Poisson time-independent probability multiplied by a probability gain, where the latter is a weight-average of the fault-section gains \((p_s^{BPT} / p_s^{Pois})\), with the weights being section moment rates.
This methodology is not entirely self-consistent (Field and Gupta, 2007). One manifestation is that final segment probabilities listed by WGCEP (2003) and (2007), which are aggregate values computed from final rupture probabilities, are not generally equal to the segment probabilities originally computed ($P^BPT$). Another manifestation, exemplified in Figure 4, is that the distribution of segment recurrence intervals implied by Monte Carlo simulations (i.e., letting the model play out over many cycles as described by Field and Gupta (2007)) is inconsistent with that assumed, especially with respect to the presence of short recurrence intervals. The basic issue is that nothing stops a segment from rupturing by itself one day, and then being “triggered” by a neighboring segment a few days later, producing very short segment recurrence intervals that are inconsistent with the probability distribution assumed in the first place. Furthermore, the simulated rate of events is biased high relative to the original long-term rate (e.g., by about 3% for the UCERF2 example in Figure 4).

Despite these shortcomings, the methodology was applied in UCERF2 for the following reasons: an alternative was lacking; the problems were minor because UCERF2 had only a few segments per fault; and the methodology captured the overall intent of pushing probabilities in a direction consistent with elastic rebound. In other words, and from a Bayesian perspective (that probability represents a subjective degree of belief), the results were deemed preferable to applying only a time-independent Poisson model.
Figure 4. Comparison of the assumed segment recurrence interval distribution (black line) with that implied by Monte Carlos simulations (gray histograms) for the Hayward North segment, obtained using the UCERF2 model (both the long-term rate model and the time-dependent probability calculations represented by Equation (1)). The discrepancy at lowest recurrence intervals indicates a self-consistency problem. This figure was adapted from Field and Gupta (2007), which also gives a fuller description of how it was generated.

Unfortunately these problems worsen as faults are divided into shorter, more numerous segments, and especially when segmentation is relaxed altogether. Figure 5 shows Monte-Carlo simulation results when the UCERF2 methodology (equation (1)) is applied to the branch-averaged UCERF3 earthquake rate model (which is un-segmented; Figure 3). These simulations, as well as those presented below for the new methodology, proceed as follows:

1. Compute a time-dependent probability gain for each rupture at the current time, and for a forecast duration equal to one over the total long-term rate of ruptures (the average time to next event).

2. Sum the implied time-dependent rupture rates (the product of the gain and long-term rate) over all ruptures to obtain a time-dependent $TotalRate$.

3. Randomly sample the time of next event from a Poission model using $TotalRate$. 
4. Randomly sample which event has occurred at this time from the relative, time-dependent probability of each.

5. Update the date of last event on each section utilized by the sampled rupture.

6. Repeat steps 1 through 5 until a catalog of the desired duration is obtained.

Care is taken to ensure insensitivity to the initial date of last event on each section, and in terms of having a negligible probability of more than one occurrence of each rupture in each time step. The code was also tested by applying a Poisson model (i.e., setting all rupture gains to 1.0), and ensuring that all simulated rupture rates converge to the original/imposed values for adequately long simulations.

Figure 5a compares simulated versus assumed fault-section recurrence intervals, revealing a larger discrepancy than for the UCERF2 example (Figure 4). Not only are there more events occurring at short recurrence intervals, but the aperiodicity implied by the simulation is 0.56, compared to a value of 0.2 applied in computing segment probabilities. Figure 5b shows that the simulated magnitude-frequency distribution is about 20% higher than that of the long-term model. This discrepancy exists throughout the fault system, as indicated by plots of simulated versus imposed rates on each section (Figure 5c for event rates, and Figure 5d for slip rates). All of these problems stem from attempting to apply a point process model to a something that is clearly not a point process, at least not as formulated. We will return to this after exploring what physics-based earthquake simulators imply with respect to elastic-rebound predictability.
Figure 5. Monte Carlo simulation results obtained by applying the UCERF2 time-dependent probability calculations to the UCERF3 long-term earthquake rate model (where the latter represents an average over all 720 logic-tree branches for Fault Model 3.1). An aperiodicity of 0.2 was applied to all fault sections, and the simulation was run for 500,000 years (producing 265,868 events). 

(a) Comparison of the assumed (black) versus simulated (red) distribution of recurrence intervals for all fault sections, where each interval has been normalized by that expected in order to aggregate results over sections with differing long-term rates. 

(b) Simulated cumulative magnitude frequency distribution (red) versus that expected from the long-term model (black). 

(c) Simulated versus expected rate of events (participation rate) for each fault section. 

(d) Same as (c), but for slip rate.

Exploration of Physics-based Earthquake Simulators

Critics of elastic rebound have rightly pointed out that a number of debatable assumptions are required in applying the models, particularly with respect to fault segmentation, the appropriate
choice of renewal-model, and aperiodicity. Given interactive complexity of the fault system, along with our limited understanding of the physics, one could reasonably question the extent to which any simple renewal model adequately approximates earthquake recurrence. The fundamental problem, stemming from the long repeat times of large earthquakes, is a lack of observations with which to test our model assumptions.

Significant progress has fortunately been made in the development of physics-based earthquake simulators, as exemplified by a recent “focused issue” on such models in *Seismological Research Letters* (2012, Volume 83, Number 6). The preface to this publication defines these simulators as:

“computer programs that model long histories of earthquake occurrence and slip using various approximations of what is known about the physics of stress transfer due to fault slip and the rheological properties of faults” (Tullis, 2012a).

In other words, physics based simulators provide synthetic catalogs of large events that we can use, in lieu of real observations, to infer elastic rebound predictability. Although all existing simulators effectively impose some form of elastic rebound, the question is how much predictability remains given complex stress interactivity among ruptures in the fault network.

We understand that all models are an approximation of the system, and that physics-based simulators have particularly unique challenges with respect to: 1) our limited understandings of the physics; 2) computational challenges and consequent approximations with respect to modeling the physics that is assumed; 3) uncertainties in the structure of faults, especially at depth; 4) tuning the models to match observations such as paleoseismic event rates; and 5) inferring earthquake probabilities conditioned on what is know with respect to the dates of historical events (the essential focus of this paper). Due to computational demands, it is also currently impractical to propagate all the epistemic uncertainties that we suspect are important to simulator results.
We can, nevertheless, ask what current state-of-the-art simulators imply with respect to elastic-rebound predictability, keeping in mind that all simulator models could be misleading in this regard. We must also keep in mind, however, that previous WGCEPs were forced to make blind assumptions with respect to renewal models, whereas we now have the opportunity, if not an obligation, to test any assumptions against these more physical models.

Table 1 lists the four physics-base simulators presented and evaluated in the recent issue of Seismological Research Letters, together with a list of features included in each (see Tullis et al. (2012b) for details). A comparison of results is given in another paper by Tullis et al. (2012c), where they conclude that all simulators are generally consistent with observations, but that observations are too limited to fully test the models. They also conclude that “...the physics-based simulators show similar behavior even though there are large differences in the methodology... [suggesting] ... that they represent realistic behavior”. We extend their analysis here with respect to elastic-rebound predictability.

Figure 3b shows the fault model and long-term slip rates used by the physics-based simulators (Tullis et al., 2012c), where faults are divided into equal-area “elements” that are smaller than the sections used in UCERF3 (Figure 3a). As an extension of the verification analyses presented by Tullis et al. (2012c), we tested the consistency between simulated slip rates and those imposed in the first place. This revealed an indexing problem with the VIRTCAL output file, which was not fixed in time to include their results in the analysis here (although analyses of previous VIRTCAL results are consistent with conclusions drawn here).

| Attributes | of the four earthquake simulator models presented in the “focused issue” of Seismological Research Letters (2012, Volume 83, Number 6). References for each model are as follows: ALLCAL (Ward, 2012); VERTCAL (Sachs et al., 2012); RSQSim |
(Richards-Dinger and Dieterich, 2012); and ViscoSim (Pollitz, 2012). Adapted from Table 1 of Tullis et al. (2012b).

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Physics-based Simulator Model</th>
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<td></td>
<td>ALLCAL</td>
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<tr>
<td>Viscoelastic stress transfer</td>
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<tr>
<td>Layered elasticity</td>
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**Recurrence Intervals at Points on Faults**

Figure 6 shows simulator recurrence-interval histograms for ruptures at a typical location – the Pitman Canyon paleoseismic site on the San Andreas Fault (SAF). This analysis only considers “supra-seismogenic” ruptures, which are defined as those having a square-root area that exceeds the average down-dip width of the faults involved. None of the histograms in Figure 6 conform to any of the typically assumed renewal models (Figure 1). Furthermore, two of the histograms show a significant spike at the lowest bin, implying re-rupture within 15 years (~7% of the average recurrence interval), which are caused by the overlap of events that occur primarily on opposite sides of the site. These site-specific results should give us pause, not only with respect to renewal-model assumptions for a point on a fault, but also with respect to the interpretation of paleoseismic data. That is, unless faults are segmented, there seems no good reason to assume a simple renewal model should apply to a site.

![Histograms showing recurrence intervals for ALLCAL, RSQSim, and ViscoSim](image-url)
Figure 6. Recurrence-interval distributions from physics-based earthquake simulators (as labeled) at the Pitman Canyon paleoseismic site on the southern San Andreas Fault (Seitz et al., 2000). This analysis considers only supra-seismogenic ruptures, defined as having an area greater than the square of the average down-dip width. The latitude and longitude for this site is 34.25231 and -117.43028, respectively. Maximum-likelihood analysis of paleoseismic data by Biasi (2013) gives a mean recurrence interval of 173 years (with 95% confidence bounds of 106 to 284 years).

Figure 7 shows the distribution of normalized recurrence-intervals for all locations (similar to Figure 6, but averaged over all surface elements). While the results are smoother than at the individual site in Figure 6, the distributions are still inconsistent with any of the traditional renewal models, especially at the lowest recurrence-interval bins. Again, this is largely a manifestation of ruptures not honoring strict segmentation. RSQSim is unique in exhibiting a distinct spike at the lowest bin, consistent with that model’s unique inclusion of aftershock triggering (via rate and state friction); presumably earthquakes are triggering events on adjacent sections of fault, but with some overlap of their rupture surfaces.

One problem we therefore face is that recurrence intervals at a point on a fault do not conform to any of the classic renewal models, and nor should we expect them to. The other problem is that, even if we knew exactly what the recurrence-interval distribution is at each point along a fault, it is not obvious how to map these onto the probabilities of individual ruptures, especially in a way that preserves long-term event rates.
Figure 7. The distribution of simulated recurrence intervals for all surface elements, where an event is counted only if it is supra-seismogenic (as defined above). The recurrence intervals are normalized by element averages in order to make meaningful comparisons among locations with different long-term rates. \( N \) represents the number of recurrence intervals and \( \alpha \) represents the implied aperiodicity (observed COV).

**Recurrence Intervals for Ruptures**

We now explore whether we can transform the “recurrence of what” question in order to get better agreement with traditional renewal models, which could then form the basis for more self-consistent elastic-rebound probability calculations. The idea is essentially this: suppose we know exactly where the next big earthquake will occur, but we are left with figuring out the timing of when it might occur. Could we use the average date of last event, plus some expected recurrence interval for the rupture, to compute a conditional probability from a traditional renewal model?

To explore this question, let's define the average time since last event for a simulated (i.e., “observed”) rupture as

\[
\bar{T}_r = \frac{1}{N} \sum T_{er}
\]

where \( T_{er} \) is the time between the \( r^{th} \) event and the previous supra-seismogenic rupture on element \( e \), and where the sum is over all \( N \) elements utilized by the \( r^{th} \) rupture (matrix \( G_{er} \) is excluded for simplicity). The question is how these “observed” intervals compare with what we might expect. One way to compute an expected recurrence interval for the \( r^{th} \) rupture is to simply average over the long-term intervals of the elements involved:

\[
\mu_r^{RI} = \frac{1}{N} \sum \mu_e
\]

where the “\( RI \)” superscript indicates that element recurrence intervals have been averaged (in contrast to an alternative approach discussed below) and where the element intervals \( (\mu_e) \) are computed from the simulation itself (i.e., the total number of supra-seismogenic events on each
element divided by the simulation duration). We can then define observed, normalized rupture recurrence intervals as

\[ \eta_r^{RI} = \frac{T_r}{\bar{\mu}_r^{RI}} , \]

and ask whether the distribution of these values, given all simulated, supra-seismogenic ruptures, matches any of the classic renewal models.

The answer is shown in **Figure 8**, where \( \eta_r^{RI} \) histograms are given for each simulator, together with the best fits to various renewal models. The distributions for ALLCAL and RSQSim are now much more consistent with traditional renewal models, being relatively well approximated by either a BPT or Lognormal distribution with an aperiodicity of \( \sim 0.2 \) (which is significantly lower than the aperiodicity values of 0.36 to 0.47 observed at simulator sites; **Figure 7**). Note also that there are very few, if any, ruptures in the first bin, implying that supra-seismogenic ruptures do not recur quickly within the surface area of larger events. The histogram for ViscoSim is markedly different, with a mean value well above 1.0 and a higher implied aperiodicity (albeit one that still implies significant elastic-rebound predictability). Because ViscoSim is the only simulator to include viscoelastic effects (**Table 1**), this model could be indicating something important. However, this simulator is also relatively new and less tested than the others, so actionable inferences may be premature.

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**Figure 8.** The distribution of normalized rupture recurrence intervals (defined in the text) for each simulator, where the values in parentheses are the observed mean, aperiodicity, and number of events, respectively. Also shown are best-fitting renewal models (as labeled, and in a least-squares sense),
with the values in parentheses being the mean and aperiodicity, respectively. The BPT model fits are not visible because they lie directly under the Lognormal curves.

There are other reasonable approaches for computing normalized rupture recurrence intervals ($\eta_r$). One is to compute the expected recurrence interval by averaging element rates (rather than recurrence intervals), and then taking the inverse:

$$\bar{\mu}_r^{Rates} = \left[ \frac{1}{N} \sum \frac{1}{\mu_e} \right]^{-1}$$

(where the “Rates” superscript indicates that element rates are averaged). We then have the following alternative option for computing $\eta_r$:

$$\eta_r^{Rates} = \frac{T_r}{\bar{\mu}_r^{Rates}}.$$

A third option is defined by averaging the normalized time since last event on each element:

$$\eta_r^{NTS} = \frac{1}{N} \sum \eta_e = \frac{1}{N} \sum \frac{T_{er}}{\mu_e}$$

(where superscript “NTS” indicates that it is based on normalized time since last).

Analyses of simulator results indicate that both $\eta_r^{RI}$ and $\eta_r^{NTS}$ give statistically equivalent results, at least for inferences made here, but that $\eta_r^{Rates}$ produces significant biases and increased aperiodicity (decreased predictability) due to outliers. This can be illustrated with a very simple example. Suppose half of the elements for a rupture have a mean recurrence interval and time since last event of 100 years ($\mu_e = 100$ and $T_{er}=100$), and the other half have values of 10,000 years for both $\mu_e$ and $T_{er}$. The equations above yield $T_r=550$, $\bar{\mu}_r^{RI}=550$, and $\bar{\mu}_r^{Rates}=198$, and therefore normalized recurrence interval values of 1.0 for both $\eta_r^{RI}$ and $\eta_r^{NTS}$. However, $\eta_r^{Rates}$ has value of 25.5, which is clearly inconsistent with traditional renewal models. Arithmetic averages are pulled high by any relatively large values. Because an element with an anomalously large time since last event must also have a large recurrence interval (the two are correlated), the influence generally cancels for $\eta_r^{RI}$ and $\eta_r^{NTS}$ because the numerator and denominator are effected more equally, whereas $\eta_r^{Rates}$ can be pushed to extreme values.
Simulator analyses presented previously (Field, 2011) implied a significant degree of both time- and slip-predictability (Shimazaki and Nakata, 1980). That is, deviations in normalized rupture recurrence intervals appeared to correlate with both the average slip in the last event (time-predictability) and with the average slip of the next event (slip predictability). These apparent correlations, however, turned out to be an artifact of not properly accounting for variation in slip rate among faults. Figure 9 shows the updated/corrected result, implying a lack of both time and slip predictability in the simulators, at least in terms of anything worth accounting for.

The renewal-model fits shown in Figure 8 include all supra-seismogenic events, and therefore hid any magnitude dependence. Table 2 lists aperiodicity values obtained for three separate magnitude ranges, indicating a significant decrease for larger events. That smaller events have greater aperiodicity is intuitively consistent with such events being more susceptible to evolving stress heterogeneities.

![Figure 9. a) Scatter plot of normalized rupture recurrence intervals versus the normalized average slip in last event (where the latter normalization is based on the average long-term slip of each element).](image-url)
The lack of correlation implies negligible time-predictability (Shimazaki and Nakata, 1980), where larger previous slips would produce longer repeat times. b) Same as (a), but where the slip is for the next, rather than last, event; lack of correlation here implies negligible slip-predictability as well (Shimazaki and Nakata, 1980), where longer repeat times would have produced larger next events.

In summary, the simulator analysis here indicates that recurrence intervals at points on faults are not well matched by traditional renewal models. However, by changing the focus to normalized rupture recurrence intervals ($\eta_r$), we see much more consistency. If we were privy to the location of the next large rupture, then we could compute its conditional probability of occurrence using the renewal-model fits in Figure 8 (and including the magnitude-dependent aperiodicity values in Table 2). Of course we do not know which event will occur next, so the following section presents an algorithm for computing probabilities when there are several overlapping rupture possibilities.

| Table 2. Aperiodicity values inferred from physics-based simulators for different magnitude ranges. These are for $\eta_r^{NTS}$, but results for $\eta_r^{TS}$ are statistically equivalent. |
|----------------------------------|-----------------|-----------------|-----------------|
|                                  | $M \leq 6.7$    | $6.7 < M \leq 7.7$ | $M > 7.7$ |
| ALLCAL                           | 0.29            | 0.20             | 0.12          |
| RSQSim                           | 0.42            | 0.21             | 0.09          |
| ViscoSim                         | 0.51            | 0.49             | 0.09          |

**UCERF3 Methodology**

Our goal here is to derive a less biased and more self-consistent approach for computing conditional time-dependent probabilities on an un-segmented fault or fault system. Building on the physics-based simulator results above, we first assume that the $r^{th}$ rupture will be the next event to occur, and then compute its expected recurrence interval as a weight average over the long-term recurrence intervals ($\mu_s$) of the sections involved:
\[ \mu_{r}^{\text{cond}} = \frac{\sum_{s} \mu_{s} A_{s}}{\sum A_{s}}. \]

Here, \( A_{s} \) is section area, and the sums are only over the sections utilized by the \( r^{th} \) rupture (matrix \( G_{sr} \) is left out). The superscript \( \text{cond} \) indicates that the expected recurrence interval is conditioned on the fact that the rupture will be the next event to occur. The use of section area for weights is consistent with what would be obtained by dividing our fault sections into smaller, equal-area elements (equivalent to the simulator analysis above).

Likewise, we define the average normalized time since last event for each rupture as

\[ \eta_{r} = \frac{\sum (T_{s}/\mu_{s}) A_{s}}{\sum A_{s}} \]

where \( T_{s} \) is the time since last event on the \( s^{th} \) section, and the sums are only over the sections utilized by the \( r^{th} \) rupture. As discussed in the simulator analysis above, there are alternative ways one can compute both \( \mu_{r}^{\text{cond}} \) and \( \eta_{r} \); the practical implications of these are addressed in the Discussion section below.

From the average normalized time since last event (\( \eta_{r} \)), the normalized forecast duration (\( \Delta T/\mu_{r}^{\text{cond}} \)), and an assumed aperiodicity (\( \alpha \)), we can now compute the conditional probability for the rupture using a renewal model. Following the two most recent WGCEPs, and with support from the simulator analysis above, we use the BPT renewal model to compute conditional rupture probabilities, which we write as \( P_{r}^{\text{BPT}} \). Note that this probability is “conditional” in both the traditional sense (that there has been a specified period of time since the last event), and in the sense that the \( r^{th} \) rupture is assumed to be the next supra-seismogenic event to occur.

Recall that the long-term, non-conditional recurrence interval of a given rupture (\( \mu_{r} \)) is generally much larger than \( \mu_{r}^{\text{cond}} \) due to overlapping rupture possibilities (e.g., events that have slightly different endpoints). In fact, \( \mu_{r} \) depends directly on the level of fault discretization, with values approaching infinity in the continuum limit. Furthermore, the ratio of the two values
(\(\mu_r^{\text{cond}} / \mu_r\)) approximates the fraction of time, on average, that the \(r^{th}\) rupture is the event to occur (as opposed to one of the other overlapping rupture possibilities). This fraction is precisely what we need to account for the fact that we lack information on which event will actually occur next. Specifically, we compute the net occurrence probability for each rupture as:

\[
P_r^{U3} = P_r^{BPT} \left[ \frac{\mu_r^{\text{cond}}}{\mu_r} \right]
\]  

(2)

In words, the first term (\(P_r^{BPT}\)) represents the conditional probability assuming the \(r^{th}\) rupture occurs next, and the second term (\(\mu_r^{\text{cond}} / \mu_r\)) is a proxy for the probability that the \(r^{th}\) rupture is “chosen” given an occurrence of one of the overlapping ruptures. Note also that Equation (2) reverts to the appropriate form for a completely segmented model (where \(\mu_r^{\text{cond}} = \mu_r\)). The superscript “U3” is applied in Equation (2) to indicate the developed for, and use in, UCERF3.

This approach represents each possible rupture with a separate BPT renewal model, with a high degree of model-parameter correlation to the extent that ruptures overlap. Having learned that seeming reasonable approaches can exhibit important biases when subjected to Monte Carlo simulations (e.g., Figure 5), Equation (2) has been put through extensive tests, a small subset of which are presented here.

Figure 10 shows Monte Carlo simulation results obtained by applying \(P_r^{U3}\) to the UCERF3 long-term model (like Figure 5, but using Equation (2) rather than Equation (1)), and assuming a magnitude-independent aperiodicity of 0.2. The fact that the average fault-section recurrence-interval histogram (Figure 10a) does not agree with any of the classic renewal models is not a problem here, as we have made no explicit assumptions about this distribution. The percentage of points in the shortest-interval bin is less than for \(P_r^{U2}\) (Figure 5a), and generally more consistent with physics-based simulator results (Figure 7). The simulated versus imposed section event rates and slip rates show better agreement for \(P_r^{U3}\) (Figure 10c and 10d) compared
to Figure 5c and 5d for $P_r^{U2}$, respectively). The same can generally be said about cumulative magnitude frequency distributions (Figure 10b versus Figure 5b); however, the $P_r^{U3}$ results do exhibit a $\sim$15% over-prediction at the lowest magnitudes, as well as a similar under-prediction at highest magnitudes, indicating that the UCERF3 approach is not perfect either. Finally, Figure 10e compares the simulated versus assumed/imposed normalized rupture recurrence intervals, indicating much more self-consistency than in the UCERF2 approach (Figure 5a).
Figure 10. Same as Figure 5 (including an aperiodicity of 0.2 for all magnitudes), but applying the UCERF3 time-dependent probability calculation ($P_r^{UCF}$ of Equation (2)) rather than the UCERF2 calculation ($P_r^{UCF_2}$ of Equation (1)). e) Comparison of simulated (red bins) and assumed/imposed (black line) normalized rupture recurrence intervals.
The UCERF2 time-dependent model used the following three aperiodicity values: 0.3, 0.5, and 0.7, with logic-tree weights of 0.2, 0.5, and 0.3, respectively. These values are higher than the aperiodicity of 0.2 applied in the Monte Carlo simulations here (Figure 5 and Figure 10). However, note that the final aperiodicity implied for fault sections is higher than that imposed, being ~0.56 for the UCERF2 approach (Figure 5a) and ~0.48 for the UCERF3 approach (Figure 10a). Thus, applying an aperiodicity of 0.2 at the rupture level produces fault-section values that are in general agreement with the UCERF2 preferred value of 0.5. That normalized rupture recurrence intervals have a lower aperiodicity than the fault sections is consistent with the simulator results above (Figure 7 versus Figure 8).

Table 3 list three sets of magnitude-dependent aperiodicity values that we also tested. The intent is for these to represent a reasonable range of options, guided in part be the simulator results above (Table 2). Monte Carlo simulation results for the “MID-RANGE” values are shown in Figure 12, where observed normalized rupture recurrence intervals are seen to match the target distributions well for each magnitude range (Figure 11c-f). There is less of a discrepancy between the target and simulated magnitude-frequency distributions at low magnitudes (~6%, down from ~15% in Figure 10). However, the discrepancy has increased to ~10% for M≥6.7, and we still have an under prediction at the highest magnitudes. Total section event rates and slip rates (not shown) look equivalent to those for the magnitude-independent aperiodicity case (Figure 10). Table 3 summarizes the total rate biases for this and the other two sets of aperiodicity options, with the most salient result being an over-prediction of total M≥6.7 event rates by ~10%, on average.
Figure 12. Monte Carlo simulation results for the UCERF3 time-dependent probability calculation \(P_{\text{rupt}}^{\text{t}}\) of Equation (2)] using the “MID RANGE” magnitude-dependent aperiodicity values listed in Table 3.  

(a) Distribution of simulated recurrence intervals for all fault sections (as in Figure 5).  

(b) Simulated cumulative magnitude frequency distribution versus that imposed (from the long-term model).  

c-f) Comparison of simulated (red bins) and assumed/imposed (black line) normalized rupture recurrence intervals for each magnitude range.
Table 3. Magnitude-dependent aperiodicity options (first column) used for the UCERF3 probability calculations, together with various observed/target rate ratios from the Monte Carlo simulations described in the text.

<table>
<thead>
<tr>
<th>Mag-Dependent Aperiodicity</th>
<th>Total Rate Ratio</th>
<th>M≥6.7 Rate Ratio</th>
<th>Total Moment Rate Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>M≤6.7, 6.7&lt;M≤7.2, 7.2&lt;M≤7.7, M&gt;7.7</td>
<td>1.06</td>
<td>1.09</td>
<td>0.94</td>
</tr>
<tr>
<td>LOW RANGE: 0.4, 0.3, 0.2, 0.1</td>
<td>1.05</td>
<td>1.11</td>
<td>1.03</td>
</tr>
<tr>
<td>MID RANGE: 0.5, 0.4, 0.3, 0.2</td>
<td>1.06</td>
<td>1.11</td>
<td>1.12</td>
</tr>
<tr>
<td>HIGH RANGE: 0.6, 0.5, 0.4, 0.3</td>
<td>1.10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average Ratio:</td>
<td>1.06</td>
<td>1.10</td>
<td>1.03</td>
</tr>
</tbody>
</table>

Discussion

While the elastic-rebound probability calculation proposed here \( P_r^{U3} \), Equation (2)) represents a clear improvement over that applied previously \( P_r^{U2} \), Equation (1)), Monte Carlo simulations reveal a \(~10\%\) over-prediction of M≥6.7 event rates, which could have important practical implications (even though it is well below the \(~20\%\) discrepancy implied by the previous approach). The first question is whether this discrepancy is unique to any particular faults. The plots of simulated versus imposed section rates (Figure 10 c and d) imply the answer is no. Analyses also imply the same can be said of individual ruptures (none seem to be particularly biased), although such tests are compute-time limited when it comes to sampling the most infrequent model ruptures.

A simple “fix” for this rate bias is to scale all conditional recurrence intervals \( \mu_r^{cond} \) by a factor of \(~1.1\) before computing rupture probabilities (confirmed via Monte Carlo simulations). While the need for such a correction is admittedly inelegant, there is also no obvious reason why the discrepancy should be zero. After all, we are trying to model a complex earthquake system with a large set of relatively simple, separate, and somewhat correlated renewal models. Whether or not to apply this correction should be made on a case-by-case basis, especially in the context of
overall epistemic uncertainties (which in general will be larger). If the correction is applied, tests should also be conducted to confirm that our value (1.1) is applicable elsewhere, as some variation was found among tests using simplified fault models.

In computing normalized rupture recurrence intervals, another question is whether having alternative section-averaging schemes is of practical importance. As outlined in the simulator analysis above, we can either average section recurrence intervals ($\mu_s$), or section rates ($1/\mu_s$), when computing conditional rupture recurrence intervals ($\mu_t^{\text{cond}}$). Likewise, we can either average the time since last event ($T_s$), or the normalized time since last event ($T_s/\mu_t^{\text{cond}}$). An extensive set of Monte Carlo simulations was conducted to ascertain the efficacy of each option. Besides the one problematic case exemplified in the simulator section (that combining average section rates with average time since last event leads to unreasonable outliers), we found no basis for rejecting any of the remaining options; all reproduce long-term event rates, slip rates, and rupture rates equally (and somewhat surprisingly) well.

Nevertheless, the influence of averaging scheme should probably be evaluated on a case-by-case basis. For example, one situation where more important differences have been found is when the date of last event is unknown in some areas, which is currently the case for most UCERF3 fault sections. Here, one must account for all the last-event-date possibilities, with the algorithm adopted for UCERF3 (Field et al., ADD REFERENCE TO MAIN REPORT) being conceptually inconsistent with averaging dates of last events (as opposed to averaging normalized values).

The fundamental difference between the previous approach ($P_t^{U2}$, Equation (1)) and that presented here ($P_t^{U3}$, Equation (2)), is that the former averages section probabilities and the latter averages section recurrence intervals and normalized times since last event. The Monte Carlo simulations reveal that the previous approach has a greater proportion of short recurrence intervals on fault sections (e.g., about 5% of those in Figure 5a are in the first bin, meaning a
repeat time that is less than 10% of the average recurrence interval, compared to 2.5% being in the first bin of Figure 10a). These short recurrence intervals are produced by either some overlap of ruptures on adjacent sections of a fault, or by the second event being significantly larger and completely encompassing the first rupture.

Figure 13 shows examples of the spatial distribution of Monte-Carlo simulated events on the Northern SAF, for both the UCERF2 and UCERF3 time-dependent calculations and for a Poisson time-independent model. The black crosses indicate areas that quickly re-ruptured (having a repeat time less than 10% of the long-term average recurrence interval), with the UCERF2 approach having a bit more such overlap. As expected, both the UCERF2 and UCERF3 time dependent calculations produce events that are more equally spaced in time compared to the Poisson model. In fact, one could question whether the gaps and clusters produced by the Poisson model are even physically reasonable given implied stresses.

Interestingly, the degree of overlap among adjacent ruptures is directly influenced by how the probabilities are computed. For example, consider a case where all fault section areas and slip rates are identical (to remove some of the relatively minor differences between the two approaches), and where all sections have a mean recurrence interval of 100 years ($\mu_s$). If half the sections have just ruptured, and the other half have gone exactly one recurrence interval since the last event, then what is the probability of having another full-fault rupture in the near future? For $P_{r2}$ we get an implied probability gain of about 2.1 relative to the long-term probability for such an event (for a 1-year forecast and assuming an aperiodicity of 0.2). From $P_{r3}$, on the other hand, we get an implied gain of ~0.013, about a factor of ~160 less than that from $P_{r2}$. In fact, the $P_{r2}$ probability in this example is just half of what it would be if all sections have gone an average recurrence interval since the last event. If half of a large fault has just ruptured, do we really think the near-term probability of the whole fault rupturing has only gone down by 50%?
The degree of short-interval rerupturing implied by UCERF3 exhibits more consistency with the physics-based-simulators (the first bins of Figure 7 are closer to those of Figure 10a than to those in Figure 5a). However, we do not have good observational constraints on the amount of slip-surface overlap among large, closely timed ruptures on a fault (e.g., like the post-1939 sequence on the North Anatolian fault; Stein et al., 1997). Furthermore, modeling this overlap correctly will require including spatiotemporal clustering, as the distance decay of aftershock statistics alone favors triggering events on adjacent sections of a fault.
**Figure 13.** Monte-Carlo simulated events along the Northern San Andreas Fault for the Poisson model and the UCERF2 (U2) and UCERF3 (U3) probability calculations (Equations (1) and (2), respectively). The lowest (left-most) latitude corresponds to the southern end of the Santa Cruz Mts section, and the highest (right-most) latitude corresponds to the northern end of the Offshore section. Black crosses indicate repeat times that are less than one-tenth the average recurrence interval at each point.

A potentially important implication of \( P_f^{U3} \), which is also true of \( P_f^{U2} \) and other previous WGCEP models, is a zero probability for having a large supra-seismogenic rupture shortly after, and completely inside the area, of a larger event (e.g., the first bin is zero in Figure 10e). While this is consistent with the physics based simulators (Figure 8), and we know of no observational exceptions, it could nevertheless be wrong, especially when it comes to triggered events. That an earthquake might dynamically reload a central portion of a long rupture is physically reasonable, although apparently very rare.

Our use of physics-based simulators to support modeling assumptions is a first among WGCEPs, and is analogous to the use of 3D waveform modeling to guide the functional form of empirical ground-motion-prediction equations (BEST REFERENCE?). That all simulators imply a significant degree of elastic-rebound predictability might not be surprising given it is built into the models. Perhaps the more informative statement is that system-level stress interactions do not appear to overwhelm, or erase, elastic-rebound predictability. The obvious challenge to any enduring critics of elastic rebound would therefore be to create a simulator that does erode all such predictability, while providing at least an equally good match to observational constraints.

The utilization of Monte Carlo simulations to evaluate models might appear unnecessary to those only interested in next-event probabilities. However, one of our goals is to eventually deploy a real-time operational earthquake forecast that includes spatiotemporal clustering (Jordan and Jones, 2010; Jordan et al., 2011). Preliminary studies imply that statistical clustering models will not work without including some form of elastic rebound (Field, 2012) and that useful OEFs will most likely be simulation based (i.e., providing suites of synthetic catalogs). The Monte
Carlos simulations presented here may therefore constitute an important step toward achieving an OEF for California.

**Conclusions**

The methodology proposed here for computing time-dependent, elastic-rebound-based earthquake probabilities for un-segmented fault models is more self-consistent and less biased than the previous approach, and agrees well with the overall behavior found in the latest generation of physics-based earthquake simulators. An additional advantage is the ability to implement magnitude-dependent aperiodicities, which are both physically appealing and evidenced by the physics-based simulators as well. The approach is not perfect, however, in terms of honoring all long-term event rates exactly, although this bias can easily be corrected if one so desires.

None of this guarantees that new approach is consistent with nature, as we presently lack adequate observations to test such models. The real earthquake system is complex, and may not lend itself perfectly to any simple models. What this paper provides is a rational way of including elastic rebound in un-segmented fault models, but only after assuming such behavior exists in the first place. An additional question for practitioners is whether an adequate range of epistemic uncertainties is spanned by this new approach (e.g., should alternatives to the BPT renewal model be applied?). Finally, one of the physics-based simulators (ViscoSim; Pollitz, 2012) suggests that viscoelasticity may have an important influence on earthquake recurrence, which we have not fully explored here.
Data and Resources

The physics-based-simulator data were obtained from the respective authors. All other calculations and analyses were conducted using OpenSHA (http://www.OpenSHA.org), which uses JFreeChart (http://www.jfree.org/jfreechart) for generating the types of plots shown here.

Acknowledgements

Both Thomas Jordan and Kevin Milner provided valuable participatory reviews, with questions from the former leading to the inclusion of magnitude-dependent aperiodicity and further discussion of the fault-section-averaging options, and questions from the latter resulting in an important modification of Equation (2), where an earlier form was not applicable to long-duration forecasts. Warner Marzocchi also provided valuable feedback on a much earlier version of this paper (and on behalf of the UCERF3 Scientific Review Panel, chaired by William Ellsworth).

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References


Appendix

This appendix gives the probability density functions of various renewal models in terms of the mean and aperiodicity of recurrence intervals, where aperiodicity is defined as the standard deviation divided by the mean. Except where otherwise noted, the initial probability density functions cited here are widely available (e.g., Wikipedia). The functional forms here also assume that recurrence intervals are, by definition, greater than or equal to zero.

Lognormal Distribution

The probability density function is given as:

$$f(t; m, \sigma) = \frac{1}{t \sigma \sqrt{2\pi}} e^{-\frac{(\ln(t) - m)^2}{\sigma^2}}$$

where $m$ and $\sigma$ are the mean and standard deviation of the natural logarithm of recurrence intervals. The mean of the distribution is given as

$$\mu = e^{m + \sigma^2/2}$$

and the variance is

$$(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$$

Therefore, the aperiodicity (square-root of variance divided by the mean) can be written as

$$\alpha = \sqrt{\frac{(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}}{e^{m + \sigma^2/2}}} = (e^{\sigma^2} - 1)^{0.5}$$

Solving this for $\sigma$ in terms of aperiodicity we have

$$\sigma = \sqrt{\ln(\alpha^2 + 1)}$$

Substituting and rearranging we get the probability density as a function of $\mu$ and $\alpha$:

$$f(t; \mu, \alpha) = \frac{1}{t \sqrt{2\pi \ln(\alpha^2 + 1)}} e^{-\frac{[\ln(t) - \ln(\mu) - 0.5 \ln(\alpha^2 + 1)]^2}{\ln(\alpha^2 + 1)}}$$
Brownian Passage Time (BPT)

The BPT distribution is also referred to as an Inverse Gaussian or Wald distribution in the statistics literature. Matthews et al. (2002) give the probability density function in terms of $\mu$ and $\alpha$ as:

$$f(t; \mu, \alpha) = \left(\frac{\mu^2}{2\pi \alpha^2 t^3}\right)^{1/2} e^{-\frac{(t-\mu)^2}{2\mu \alpha^2 t}}$$

(Their equation (12)).

Weibull Distribution

The probability density function of a Weibull distribution is:

$$f(t; \lambda, k) = \frac{k}{\lambda} \left(\frac{t}{\lambda}\right)^{k-1} e^{-(t/\lambda)^k}$$

where $\lambda$ and $k$ are a scale and shape parameter, respectively, both of which are positive. The mean is given as:

$$\mu = \lambda \Gamma(1 + 1/k)$$

and the variance is given as:

$$\lambda^2 \Gamma(1 + 2/k) - \mu^2$$

where $\Gamma$ is the gamma function. Aperiodicity is then given as the square root of the variance divided by the mean:

$$\alpha = \frac{\sqrt{\lambda^2 \Gamma(1 + 2/k) - \Gamma(1 + 1/k)^2}}{\lambda \Gamma(1 + 1/k)} = \frac{\sqrt{\Gamma(1 + 2/k) - [\Gamma(1 + 1/k)]^2}}{\Gamma(1 + 1/k)}$$

Note that the scale parameter drops out on the right hand side, leaving $\alpha$ dependent only on the shape parameter $k$. Due to the gamma function dependence, we solve for the shape parameter numerically when given a desired aperiodicity (e.g., see the following OpenSHA method: org.opensha.sha.earthquake.calc.recurInterval.WeibullDistCalc.getShapeParameter(aper)).
**Exponential Distribution**

The Exponential distribution, which describes the inter-event times for a Poisson process, has the following probability density function:

\[
f(t; \mu) = \mu^{-1} e^{-t/\mu}
\]

where \( \mu \) is the mean. The standard deviation of this distribution is equal to the mean, so the aperiodicity is always 1.0.